编程语言的设计原理

Design Principles of Programming Languages

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Peking University, Spring, 2022

Recap

Simply typed lambda calculus

$\lambda_{\to}$

Syntax

$\mathtt{t}$	::=		*terms:*
		$\mathtt{x}$	*variable*
		$\lambda\mathtt{x}{:}\mathtt{T}.\mathtt{t}$	*abstraction*
		$\mathtt{t}\ \mathtt{t}$	*application*
$\mathtt{v}$	::=		*values:*
		$\lambda\mathtt{x}{:}\mathtt{T}.\mathtt{t}$	*abstraction value*
$\mathtt{T}$	::=		*types:*
		$\mathtt{T}\to\mathtt{T}$	*type of functions*
Γ	::=		*contexts:*
		$\varnothing$	*empty context*
		$\Gamma, \mathtt{x}{:}\mathtt{T}$	*term variable binding*

Evaluation $\boxed{\mathtt{t} \longrightarrow \mathtt{t}'}$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{t}_1\ \mathtt{t}_2 \longrightarrow \mathtt{t}_1'\ \mathtt{t}_2} \qquad \text{(E-APP1)}$$

$$\frac{\mathtt{t}_2 \longrightarrow \mathtt{t}_2'}{\mathtt{v}_1\ \mathtt{t}_2 \longrightarrow \mathtt{v}_1\ \mathtt{t}_2'} \qquad \text{(E-APP2)}$$

$$(\lambda\mathtt{x}{:}\mathtt{T}_{11}.\mathtt{t}_{12})\ \mathtt{v}_2 \longrightarrow [\mathtt{x} \mapsto \mathtt{v}_2]\mathtt{t}_{12} \qquad \text{(E-APPABS)}$$

Typing $\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$

$$\frac{\mathtt{x}{:}\mathtt{T} \in \Gamma}{\Gamma \vdash \mathtt{x} : \mathtt{T}} \qquad \text{(T-VAR)}$$

$$\frac{\Gamma, \mathtt{x}{:}\mathtt{T}_1 \vdash \mathtt{t}_2 : \mathtt{T}_2}{\Gamma \vdash \lambda\mathtt{x}{:}\mathtt{T}_1.\mathtt{t}_2 : \mathtt{T}_1\to\mathtt{T}_2} \qquad \text{(T-ABS)}$$

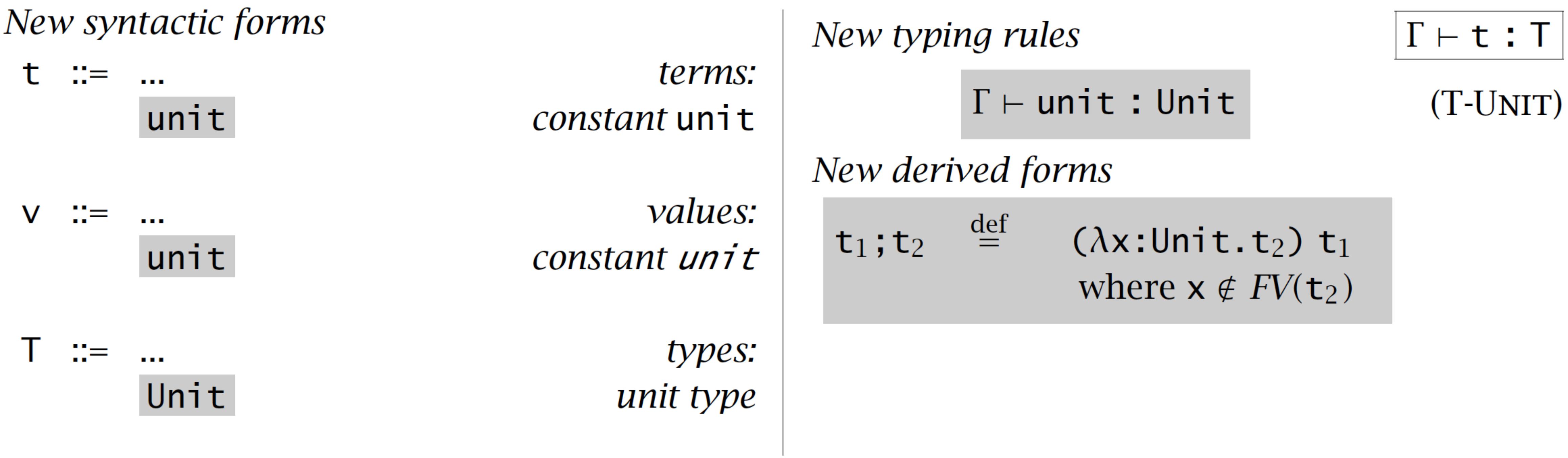
$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_{11}\to\mathtt{T}_{12} \qquad \Gamma \vdash \mathtt{t}_2 : \mathtt{T}_{11}}{\Gamma \vdash \mathtt{t}_1\ \mathtt{t}_2 : \mathtt{T}_{12}} \qquad \text{(T-APP)}$$

Assume:

all variables in Γ are different via renaming/internal

The Unit Type

- It is the singleton type (like void in C).



New syntactic forms

`t`	::=	...	*terms:*
		`unit`	*constant* `unit`
`v`	::=	...	*values:*
		`unit`	*constant* `unit`
`T`	::=	...	*types:*
		`Unit`	*unit type*

New typing rules $\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$

$$\Gamma \vdash \mathtt{unit} : \mathtt{Unit} \qquad \text{(T-UNIT)}$$

New derived forms

$$\mathtt{t}_1\mathtt{;t}_2 \overset{\text{def}}{=} (\lambda \mathtt{x}:\mathtt{Unit}.\mathtt{t}_2)\ \mathtt{t}_1 \quad \text{where } \mathtt{x} \notin FV(\mathtt{t}_2)$$

- Application: Unit-type expressions care more about “side effects” rather than “results”.

Derived Form: Sequencing $t_1 ; t_2$

- A direct extension λ^E
  - t ::= …

    $t_1; \ t_2$
  - New evaluation relation rules

$$\frac{\texttt{t}_1 \longrightarrow \texttt{t}_1'}{\texttt{t}_1\texttt{;}\texttt{t}_2 \longrightarrow \texttt{t}_1'\texttt{;}\texttt{t}_2} \qquad \text{(E-SEQ)}$$

$$\texttt{unit;}\texttt{t}_2 \longrightarrow \texttt{t}_2 \qquad \text{(E-SEQNEXT)}$$

  - New typing rules

$$\frac{\Gamma \vdash \texttt{t}_1 : \texttt{Unit} \qquad \Gamma \vdash \texttt{t}_2 : \texttt{T}_2}{\Gamma \vdash \texttt{t}_1\texttt{;}\texttt{t}_2 : \texttt{T}_2} \qquad \text{(T-SEQ)}$$

Ascription

New syntactic forms

$$\texttt{t} ::= \ldots \qquad\qquad \textit{terms}$$

$$\texttt{t as T} \qquad\qquad \textit{ascription}$$

New evaluation rules

$$\texttt{v}_1 \texttt{ as T} \longrightarrow \texttt{v}_1 \qquad \text{(E-ASCRIBE)}$$

$$\frac{\texttt{t}_1 \longrightarrow \texttt{t}_1'}{\texttt{t}_1 \texttt{ as T} \longrightarrow \texttt{t}_1' \texttt{ as T}} \qquad \text{(E-ASCRIBE1)}$$

New typing rules

$$\frac{\Gamma \vdash \texttt{t}_1 : \texttt{T}}{\Gamma \vdash \texttt{t}_1 \texttt{ as T} : \texttt{T}} \qquad \text{(T-ASCRIBE)}$$

verification

Ascription as a derived form

$$\texttt{t as T} \overset{\text{def}}{=} (\lambda \texttt{x:T. x})\ \texttt{t}$$

Let Bindings

- To give names to some of its subexpressions.

New syntactic forms

$$\mathtt{t} ::= \ldots \qquad \textit{terms}$$

$$\mathtt{let}\ \mathtt{x}\mathtt{=}\mathtt{t}\ \mathtt{in}\ \mathtt{t} \qquad \textit{let binding}$$

New evaluation rules

$$\mathtt{let}\ \mathtt{x}\mathtt{=}\mathtt{v}_1\ \mathtt{in}\ \mathtt{t}_2 \longrightarrow [\mathtt{x} \mapsto \mathtt{v}_1]\mathtt{t}_2 \qquad \text{(E-LetV)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{let}\ \mathtt{x}\mathtt{=}\mathtt{t}_1\ \mathtt{in}\ \mathtt{t}_2 \longrightarrow \mathtt{let}\ \mathtt{x}\mathtt{=}\mathtt{t}_1'\ \mathtt{in}\ \mathtt{t}_2} \qquad \text{(E-Let)}$$

New typing rules

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_1 \qquad \Gamma, \mathtt{x}{:}\mathtt{T}_1 \vdash \mathtt{t}_2 : \mathtt{T}_2}{\Gamma \vdash \mathtt{let}\ \mathtt{x}\mathtt{=}\mathtt{t}_1\ \mathtt{in}\ \mathtt{t}_2 : \mathtt{T}_2} \qquad \text{(T-Let)}$$

Records

New syntactic forms

$\mathtt{t} ::= \ldots$ *terms:*

$\{\mathtt{l}_i\mathtt{=t}_i\ ^{i\in 1..n}\}$ *record*

$\mathtt{t.l}$ *projection*

$\mathtt{v} ::= \ldots$ *values:*

$\{\mathtt{l}_i\mathtt{=v}_i\ ^{i\in 1..n}\}$ *record value*

$\mathtt{T} ::= \ldots$ *types:*

$\{\mathtt{l}_i\mathtt{:T}_i\ ^{i\in 1..n}\}$ *type of records*

New evaluation rules $\boxed{\mathtt{t} \longrightarrow \mathtt{t}'}$

$$\{\mathtt{l}_i\mathtt{=v}_i\ ^{i\in 1..n}\}\mathtt{.l}_j \longrightarrow \mathtt{v}_j \qquad \text{(E-ProjRcd)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{t}_1\mathtt{.l} \longrightarrow \mathtt{t}_1'\mathtt{.l}} \qquad \text{(E-Proj)}$$

$$\frac{\mathtt{t}_j \longrightarrow \mathtt{t}_j'}{\begin{array}{l}\{\mathtt{l}_i\mathtt{=v}_i\ ^{i\in 1..j-1}\mathtt{,l}_j\mathtt{=t}_j\mathtt{,l}_k\mathtt{=t}_k\ ^{k\in j+1..n}\}\\ \longrightarrow \{\mathtt{l}_i\mathtt{=v}_i\ ^{i\in 1..j-1}\mathtt{,l}_j\mathtt{=t}_j'\mathtt{,l}_k\mathtt{=t}_k\ ^{k\in j+1..n}\}\end{array}} \qquad \text{(E-Rcd)}$$

New typing rules $\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$

$$\frac{\text{for each } i \quad \Gamma \vdash \mathtt{t}_i : \mathtt{T}_i}{\Gamma \vdash \{\mathtt{l}_i\mathtt{=t}_i\ ^{i\in 1..n}\} : \{\mathtt{l}_i\mathtt{:T}_i\ ^{i\in 1..n}\}} \qquad \text{(T-Rcd)}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \{\mathtt{l}_i\mathtt{:T}_i\ ^{i\in 1..n}\}}{\Gamma \vdash \mathtt{t}_1\mathtt{.l}_j : \mathtt{T}_j} \qquad \text{(T-Proj)}$$

Question: {partno=5524, cost=30.27} = {cost=30.27, partno=5524}?

Variants

New syntactic forms

$$\begin{array}{lll r}
\mathtt{t} & ::= & \ldots & \textit{terms:} \\
 & & \mathtt{<l=t>\ as\ T} & \textit{tagging} \\
 & & \mathtt{case\ t\ of\ <l_i=x_i>\Rightarrow t_i}\ ^{i\in 1..n} & \textit{case} \\
 & & & \\
\mathtt{T} & ::= & \ldots & \textit{types:} \\
 & & \mathtt{<l_i{:}T_i}\ ^{i\in 1..n}\mathtt{>} & \textit{type of variants}
\end{array}$$

New evaluation rules $\boxed{\mathtt{t} \longrightarrow \mathtt{t}'}$

$$\begin{array}{l}
\mathtt{case\ (<l_j=v_j>\ as\ T)\ of\ <l_i=x_i>\Rightarrow t_i}\ ^{i\in 1..n} \\
\qquad\qquad \longrightarrow [\mathtt{x_j} \mapsto \mathtt{v_j}]\mathtt{t_j}
\end{array}$$

(E-CASEVARIANT)

$$\frac{\mathtt{t_0} \longrightarrow \mathtt{t_0'}}{\begin{array}{l}\mathtt{case\ t_0\ of\ <l_i=x_i>\Rightarrow t_i}\ ^{i\in 1..n} \\ \longrightarrow \mathtt{case\ t_0'\ of\ <l_i=x_i>\Rightarrow t_i}\ ^{i\in 1..n}\end{array}} \quad \text{(E-CASE)}$$

$$\frac{\mathtt{t_i} \longrightarrow \mathtt{t_i'}}{\mathtt{<l_i=t_i>\ as\ T} \longrightarrow \mathtt{<l_i=t_i'>\ as\ T}} \quad \text{(E-VARIANT)}$$

New typing rules $\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$

$$\frac{\Gamma \vdash \mathtt{t_j} : \mathtt{T_j}}{\Gamma \vdash \mathtt{<l_j=t_j>\ as\ <l_i{:}T_i}\ ^{i\in 1..n}\mathtt{>} : \mathtt{<l_i{:}T_i}\ ^{i\in 1..n}\mathtt{>}}$$

(T-VARIANT)

$$\frac{\begin{array}{c}\Gamma \vdash \mathtt{t_0} : \mathtt{<l_i{:}T_i}\ ^{i\in 1..n}\mathtt{>} \\ \text{for each } i \quad \Gamma, \mathtt{x_i{:}T_i} \vdash \mathtt{t_i} : \mathtt{T}\end{array}}{\Gamma \vdash \mathtt{case\ t_0\ of\ <l_i=x_i>\Rightarrow t_i}\ ^{i\in 1..n} : \mathtt{T}} \quad \text{(T-CASE)}$$

General Recursions

- Introduce "fix" operator : fix f = f (fix f)
  - It cannot be defined as a derived form in simply typed lambda calculus

New syntactic forms

$\mathtt{t} ::=$	$\ldots$	*terms*
	$\mathtt{fix\ t}$	*fixed point of* $\mathtt{t}$

New evaluation rules

$$\begin{array}{l}\mathtt{fix}\ (\lambda \mathtt{x}{:}\mathtt{T_1}.\mathtt{t_2}) \\ \longrightarrow [\mathtt{x} \mapsto (\mathtt{fix}\ (\lambda \mathtt{x}{:}\mathtt{T_1}.\mathtt{t_2}))]\mathtt{t_2}\end{array} \quad \text{(E-FixBeta)}$$

$$\frac{\mathtt{t_1} \longrightarrow \mathtt{t_1'}}{\mathtt{fix\ t_1} \longrightarrow \mathtt{fix\ t_1'}} \quad \text{(E-Fix)}$$

General Recursions

New typing rules

$$\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_1 \rightarrow \mathtt{T}_1}{\Gamma \vdash \mathtt{fix}\ \mathtt{t}_1 : \mathtt{T}_1} \qquad \text{(T-FIX)}$$

A convenient form

$$\mathtt{letrec}\ \mathtt{x}{:}\mathtt{T}_1{=}\mathtt{t}_1\ \mathtt{in}\ \mathtt{t}_2 \quad \overset{\text{def}}{=} \quad \mathtt{let}\ \mathtt{x}\ =\ \mathtt{fix}\ (\lambda \mathtt{x}{:}\mathtt{T}_1.\mathtt{t}_1)\ \mathtt{in}\ \mathtt{t}_2$$

Chapter 13: Reference

Why reference

Evaluation

Typing

Store Typings

Safety

Why & What
References

Computational Effects

Also known as *side effects*.

A *function* or *expression* is said to have a **side effect** if, in addition to returning a value, it also ***modifies some state*** or has an ***observable interaction with*** calling functions or the outside world.

- modify a *global variable* or *static variable*, modify *one of its arguments*,
- *raise an exception,*
- *write* data to *a display* or file, *read* data, or
- call other *side-effecting functions*.

In the presence of side effects, a program's behavior may depend on *history*; i.e., the *order of evaluation* matters.

Computational Effects

Side effects are the *most common way* that a program *interacts with the outside world* (*people*, *file systems*, *other computers on networks*).

The degree to which *side effects are used* depends on the *programming paradigm*.

- *Imperative programming* is known for *its frequent utilization* of side effects.
- In *functional programming*, side effects are rarely used.
  - Functional languages like *Standard ML*, *Scheme* and *Scala* do not restrict side effects, but it is customary for programmers to avoid them.
  - The functional language *Haskell* expresses side effects such as I/O and other stateful computations using *monadic* actions.

Mutability

So far, what we have discussed does not yet include *side effects* .

In particular, whenever we defined function, we *never changed variables or data*. Rather, we always computed *new data*.

- E.g., the operations to *insert an item* into the data structure *didn't effect the old copy* of the data structure. Instead, we *always built a new data structure* with the item appropriately inserted.

For the most part, programming in a functional style (i.e., *without side effects*) is a "good thing" because it's *easier to reason locally about the behavior* of the program.

Mutability

Writing values into memory locations is the **fundamental mechanism** of imperative languages such as C/C++.

Mutable structures are

- required to implement many *efficient algorithms*.
- also very convenient to represent the *current state of a state machine*.

Mutability

In most programming languages, *variables are mutable* — i.e., a variable provides both

- ***a name*** that refers to a previously calculated value, and
- ***the possibility of overwriting this value*** with another (which will be referred to by the same name)

In some languages (e.g., OCaml), these features are separate:

- variables are only for naming — the binding between a variable and its value is immutable
- introduce a new class of mutable values (called *reference cells* or *references*)
  - at any given moment, a reference *holds a value* (and can be dereferenced to obtain this value)
  - *a new value* may be assigned to a reference

Basic Examples

```
#let r = ref 5
val r : int ref = {contents = 5}
//  The value of r is a reference to a cell that always contain a number.


# r:= !r +3
# !r
-: int = 8

(r:=succ(!r); !r)
```

Basic Examples

```
# let flag = ref true;;
-val flag: bool ref = {contents = true}

# if !flag then 1 else 2;;
-: int = 1
```

Reference

Basic operations

– allocation	ref (operator)
– dereferencing	!
– assignment	:=

Is there any difference between the expressions of ?

- 5 + 3;
- r: = 8;
- (r:=succ(!r); !r)
- (r:=succ(!r); (r:=succ(!r); (r:=succ(!r); !r)

sequencing

Reference

Exercise 13.1.1 :

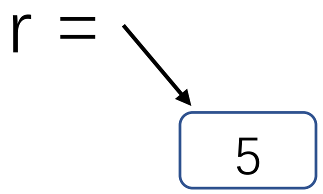
Draw a similar diagram showing the effects of evaluating the expressions

a = {ref 0, ref 0} and

b = (λx:Ref Nat. {x,x}) (ref 0)

Aliasing

A value of type ref T is a *pointer* to a cell holding a value of type T



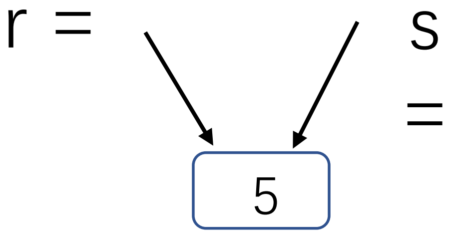
r =

5

If this value is "*copied*" by assigning it to another variable:

s=r;

the cell pointed to is not copied. (*r* and s are *aliases*)



r =

s =

5

So we can change r by assigning to s:

(s:=10; !r)

Aliasing all around us

Reference cells are *not the only language feature* that introduces the possibility of aliasing

- arrays
- communication channels
- I/O devices (disks, etc.)

The difficulties of aliasing

- The possibility of aliasing *invalidates* all sorts of useful forms of *reasoning about programs*, both *by programmers*:

  e.g., $\lambda r{:}\,Ref\ Nat.\,\lambda s{:}\,Ref\ Nat.\,(r := 2;\ s := 3;\ !\,r)$

  always returns 2 unless r and s are aliases

  and *by compilers* :

  Code motion out of loops, *common sub-expression elimination*, *allocation of variables to registers*, and *detection of uninitialized variables* all depend upon the compiler knowing *which objects a load or a store operation could reference*.
- High-performance compilers ***spend significant energy*** on *alias analysis* to try to establish when different variables cannot possibly refer to the same storage

The benefits of aliasing

The *problems of aliasing* have led some language designers simply to disallow it (e.g., Haskell).

However, there are good reasons why most languages do provide constructs involving aliasing:

- efficiency (e.g., arrays)
- shared resources (e.g., locks) in concurrent systems
- “action at a distance” (e.g., symbol tables)
- ……

Example

$c = ref\ 0$

incc $= \lambda x{:}\,Unit.\,(c \coloneqq succ(!\,c);\,!\,c)$

decc $= \lambda x{:}\,Unit.\,(c \coloneqq pred(!\,c);\,!\,c)$

incc $unit$

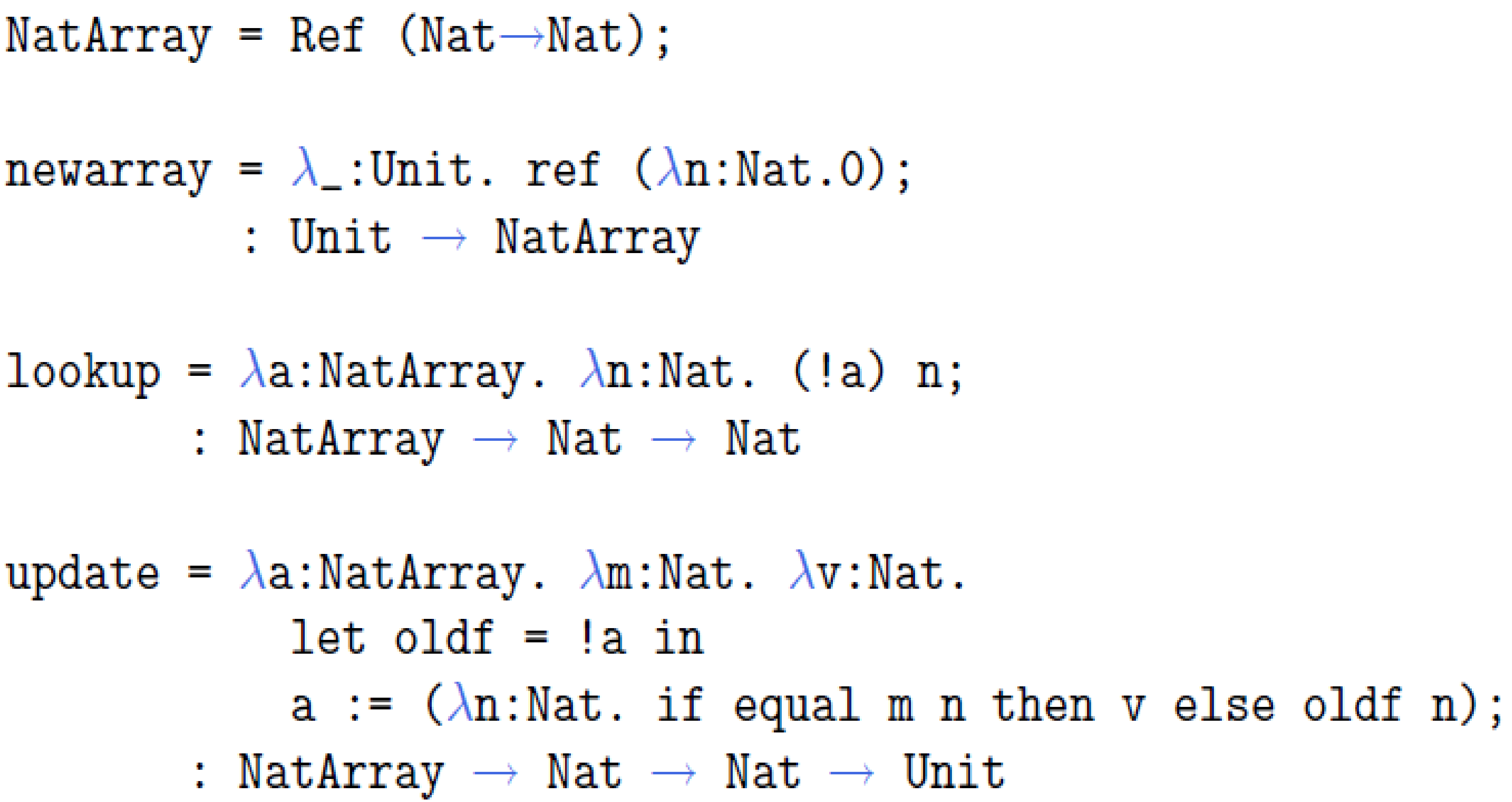
$decc\ unit$

o $= \{$i $= incc, d = decc\}$

$let\ newcounter = o$
$\lambda_{.Unit}.$
let $c\ =\ ref\ 0$ in
let incc $= \lambda x{:}\,Unit.\,(c \coloneqq succ(!\,c);\,!\,c)$ in
let decc $= \lambda x{:}\,Unit.\,(c \coloneqq pred(!\,c);\,!\,c)$
let o $= \{i = incc, d = decc\}$ in
o

Example

- Reference values of any type, including functions.



```
NatArray = Ref (Nat→Nat);

newarray = λ_:Unit. ref (λn:Nat.0);
         : Unit → NatArray

lookup = λa:NatArray. λn:Nat. (!a) n;
       : NatArray → Nat → Nat

update = λa:NatArray. λm:Nat. λv:Nat.
           let oldf = !a in
           a := (λn:Nat. if equal m n then v else oldf n);
       : NatArray → Nat → Nat → Unit
```

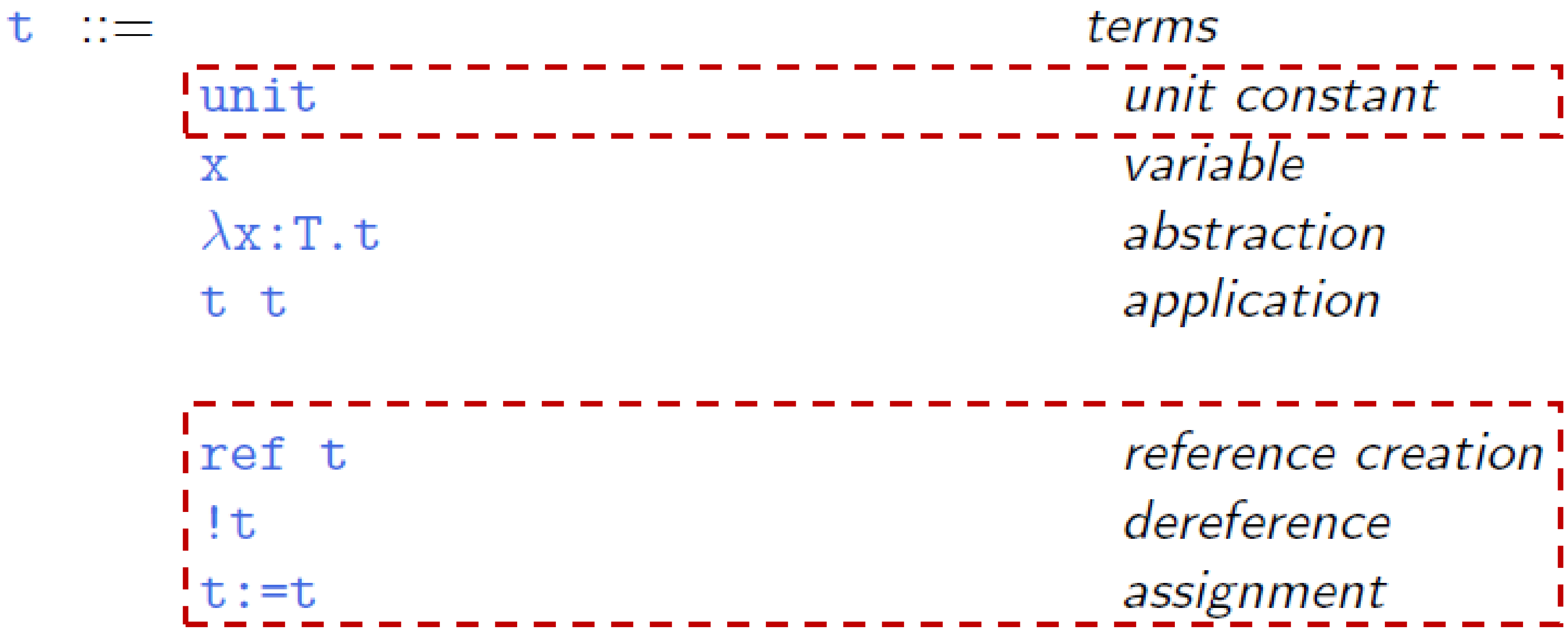
How to enrich the language

with

the new mechanism ?

Syntax

... plus other familiar types, in examples



`t`	::=		*terms*
		`unit`	*unit constant*
		`x`	*variable*
		`λx:T.t`	*abstraction*
		`t t`	*application*
		`ref t`	*reference creation*
		`!t`	*dereference*
		`t:=t`	*assignment*

Typing rules

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_1}{\Gamma \vdash \mathtt{ref}\ \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_1} \qquad (\text{T-Ref})$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_1}{\Gamma \vdash \mathtt{!t}_1 : \mathtt{T}_1} \qquad (\text{T-Deref})$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_1 \qquad \Gamma \vdash \mathtt{t}_2 : \mathtt{T}_1}{\Gamma \vdash \mathtt{t}_1\mathtt{:=}\mathtt{t}_2 : \mathtt{Unit}} \qquad (\text{T-Assign})$$

- **type system**
  - ***a set of rules*** that *assigns a property* called *type* to the various “constructs” of a computer program, such as
  - *variables*, *expressions*, *functions* or *modules*

Evaluation

What is the value of the expression ref 0 ?

Is

r = ref 0
s = ref 0

and

r = ref 0
s = r

behave the same?

Crucial observation: evaluating ref 0 must *do* something ?

Specifically, evaluating ref 0 should *allocate some storage* and yield a *reference* (or *pointer*) to that storage

So *what* is a reference?

The store

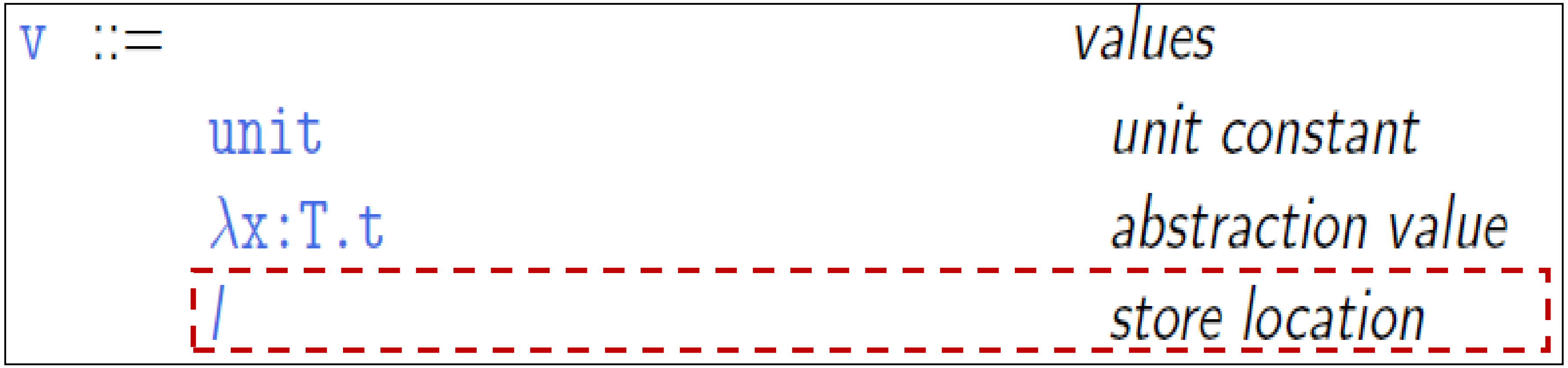
A reference names a *location* in the *store* (also known as the *heap* or just the *memory*)

What is the **store**?

- *Concretely*: an array of *8-bit bytes*, indexed by 32/64-bit integers
- *More abstractly*: an array of *values,* abstracting away from the different sizes of the runtime representations of different values
- *Even more abstractly*: a *partial function* from ***locations*** to *values*
  - set of store locations
  - Location : an abstract index into the store

Locations

Syntax of *values*:



v ::=		*values*
	unit	*unit constant*
	λx:T.t	*abstraction value*
	l	*store location*

… and since all *values* are *terms* …

Syntax of Terms

`t ::=`	*terms*
`unit`	*unit constant*
`x`	*variable*
`λx:T.t`	*abstraction*
`t t`	*application*
`ref t`	*reference creation*
`!t`	*dereference*
`t:=t`	*assignment*
l	store location

Aside

Does this mean we are going to allow programmers to *write explicit locations* in their programs??

No: This is just a modeling trick, just as intermediate results of evaluation

> Enriching the "source language" to include some *runtime structures*, we can thus continue to *formalize evaluation* as a relation between source terms

Aside: If we formalize evaluation in the *big-step style*, then we can *add locations* to *the set of values* (results of evaluation) without adding them to the set of terms

Evaluation

The *result* of *evaluating a term* now (with references)

- *depends on the store* in which it is evaluated
- *is not just a value* — we must also keep track of the *changes* that get made to the *store*

i.e., the evaluation relation should now map *a term as well as a store* to *a reduced term* and *a new store*

$$t \mid \mu \;\rightarrow\; t' \mid \mu'$$

To use the metavariable μ to *range over stores*

μ & μ' : states of the store before & after evaluation

Evaluation

A term of *the form* $\text{ref } t_1$

1. first *evaluates* inside t_1 *until it becomes a value* ...

$$\frac{\mathtt{t_1} \mid \mu \longrightarrow \mathtt{t_1'} \mid \mu'}{\mathtt{ref}\ \mathtt{t_1} \mid \mu \longrightarrow \mathtt{ref}\ \mathtt{t_1'} \mid \mu'} \qquad \text{(E-Ref)}$$

2. then *chooses* (allocates) a *fresh location* l, *augments* the store with ***a binding*** from l to v_1 , and returns l :

$$\frac{l \notin dom(\mu)}{\mathtt{ref}\ \mathtt{v_1} \mid \mu \longrightarrow l \mid (\mu, \boxed{l \mapsto \mathtt{v_1})}} \qquad \text{(E-RefV)}$$

Evaluation

A term $!t_1$ first evaluates in t_1 *until it becomes a value*...

$$\frac{\mathtt{t}_1 \mid \mu \longrightarrow \mathtt{t}_1' \mid \mu'}{!\mathtt{t}_1 \mid \mu \longrightarrow !\mathtt{t}_1' \mid \mu'} \qquad \text{(E-DEREF)}$$

... and then

1. *looks up this value* (which **must be** a ***location***, if the original term was well typed) and
2. *returns its contents* in the current store

$$\frac{\mu(l) = \mathtt{v}}{!l \mid \mu \longrightarrow \mathtt{v} \mid \mu} \qquad \text{(E-DEREFLOC)}$$

Evaluation

An assignment $t_1 \coloneqq t_2$ first evaluates t_1 and t_2 *until they become values* ...

$$\frac{\mathtt{t}_1 \mid \mu \longrightarrow \mathtt{t}_1' \mid \mu'}{\mathtt{t}_1 \mathtt{:=} \mathtt{t}_2 \mid \mu \longrightarrow \mathtt{t}_1' \mathtt{:=} \mathtt{t}_2 \mid \mu'} \qquad \text{(E-ASSIGN1)}$$

$$\frac{\mathtt{t}_2 \mid \mu \longrightarrow \mathtt{t}_2' \mid \mu'}{\mathtt{v}_1 \mathtt{:=} \mathtt{t}_2 \mid \mu \longrightarrow \mathtt{v}_1 \mathtt{:=} \mathtt{t}_2' \mid \mu'} \qquad \text{(E-ASSIGN2)}$$

... and then returns unit and updates the store:

$$l \mathtt{:=} \mathtt{v}_2 \mid \boxed{\mu} \longrightarrow \mathtt{unit} \mid \boxed{[l \mapsto \mathtt{v}_2]\mu} \qquad \text{(E-ASSIGN)}$$

Evaluation

Evaluation rules for *function abstraction* and *application* are ***augmented with stores***, but ***don’t do anything*** with them directly

$$\frac{\mathtt{t}_1 \mid \mu \longrightarrow \mathtt{t}_1' \mid \mu'}{\mathtt{t}_1\ \mathtt{t}_2 \mid \mu \longrightarrow \mathtt{t}_1'\ \mathtt{t}_2 \mid \mu'} \qquad (\textsc{E-App1})$$

$$\frac{\mathtt{t}_2 \mid \mu \longrightarrow \mathtt{t}_2' \mid \mu'}{\mathtt{v}_1\ \mathtt{t}_2 \mid \mu \longrightarrow \mathtt{v}_1\ \mathtt{t}_2' \mid \mu'} \qquad (\textsc{E-App2})$$

$$(\lambda \mathtt{x}{:}\mathtt{T}_{11}.\mathtt{t}_{12})\ \mathtt{v}_2 \mid \boxed{\mu} \longrightarrow [\mathtt{x} \mapsto \mathtt{v}_2]\mathtt{t}_{12} \mid \boxed{\mu} \qquad (\textsc{E-AppAbs})$$

Aside

Garbage Collection

Note that we are not modeling *garbage collection* — the store just *grows without bound*

It may not be problematic for most *theoretical purposes*, whereas it is clear that for *practical purposes* some form of *deallocation* of unused storage must be provided

Pointer Arithmetic

`p++;`

We can’t do any!

Store Typing

Typing Locations

Question: What is the *type of a location*?

Answer: Depends on the ***contents*** of the store!

e.g,

- in the store $(l_1 \longmapsto \text{unit}, l_2 \longmapsto \text{unit})$,
  the term $!\,l_2$ is evaluated to unit, having type $\mathbf{Unit}$
- in the store $(l_1 \longmapsto \text{unit}, l_2 \longmapsto \lambda x{:}\,\text{Unit}.\,x)$,
  the term $!\,l_2$ has type $\mathbf{Unit} \rightarrow \mathbf{Unit}$

Typing Locations — first try

Roughly, to find the type of a location l, first *look up* the current contents of l in the store, and calculate the type T_1 of the contents:

$$\frac{\Gamma \vdash \mu(l) : \mathtt{T_1}}{\Gamma \vdash l : \mathtt{Ref\ T_1}}$$

More precisely, to make ***the type of a term depend on the store*** (keeping a consistent state), we should change the *typing relation* from *three-place* to :

$$\frac{\Gamma \mid \boxed{\mu} \vdash \mu(l) : \mathtt{T_1}}{\Gamma \mid \boxed{\mu} \vdash l : \mathtt{Ref\ T_1}}$$

i.e., typing is now a *four-place relation* (about *contexts*, *stores*, *terms*, and *types*), though ***the store is a part of the context*** ……

Problems #1

However, this rule is not *completely satisfactory*, and is *rather inefficient*.

- it can make *typing derivations very large* (if a location *appears* ***many times*** in a term) !
- e.g.,

$$
\begin{aligned}
\mu = (l_1 &\mapsto \lambda x{:}\,\text{Nat}.\ 999,\\
l_2 &\mapsto \lambda x{:}\,\text{Nat}.\ (!\,l_1)\ x,\\
l_3 &\mapsto \lambda x{:}\,\text{Nat}.\ (!\,l_2)\ x,\\
l_4 &\mapsto \lambda x{:}\,\text{Nat}.\ (!\,l_3)\ x,\\
l_5 &\mapsto \lambda x{:}\,\text{Nat}.\ (!\,l_4)\ x),
\end{aligned}
$$

then how big is the typing derivation for $!\,l_5$?

Problems #2

But wait... it *gets worse* if the store contains a *cycle*.

Suppose

$$\mu = (l_1 \mapsto \lambda\text{x: Nat. } (!\, l_2)\ \text{x},\\ l_2 \mapsto \lambda\text{x: Nat. } (!\, l_1)\ \text{x})\ ,$$

how big is the typing derivation for $!\, l_2$?

Calculating a type for l_2 requires finding the type of l_1, which in turn involves l_2

Why?

What leads to the problems?

Our typing rule for locations requires us to *recalculate the type of a location every time it’s* mentioned in a term, which *should not be necessary*

In fact, once a location is first created, *the type of the initial value* is **known**, and *the type will be kept* even if the values can be changed

Store Typing

Observation:

> The typing rules we have chosen for references guarantee *that a given location* in the store is *always* used to hold *values of the same type*

These intended types can be *collected* into a ***store typing:***

— a *partial function* from *locations* to *types*

Store Typing

E.g., for

$$\begin{aligned}\mu = (l_1 &\mapsto \lambda x{:}\,\mathrm{Nat}.\ 999,\\ l_2 &\mapsto \lambda x{:}\,\mathrm{Nat}.\ (!\,l_1)\ x,\\ l_3 &\mapsto \lambda x{:}\,\mathrm{Nat}.\ (!\,l_2)\ x,\\ l_4 &\mapsto \lambda x{:}\,\mathrm{Nat}.\ (!\,l_3)\ x,\\ l_5 &\mapsto \lambda x{:}\,\mathrm{Nat}.\ (!\,l_4)\ x\ ),\end{aligned}$$

A reasonable *store typing* would be

$$\begin{aligned}\Sigma = (l_1 &\mapsto \mathtt{Nat}\rightarrow\mathtt{Nat},\\ l_2 &\mapsto \mathtt{Nat}\rightarrow\mathtt{Nat},\\ l_3 &\mapsto \mathtt{Nat}\rightarrow\mathtt{Nat},\\ l_4 &\mapsto \mathtt{Nat}\rightarrow\mathtt{Nat},\\ l_5 &\mapsto \mathtt{Nat}\rightarrow\mathtt{Nat})\end{aligned}$$

Store Typing

Now, suppose we are given *a store typing* Σ describing the store μ in which we intend to evaluate some term t

Then we can use Σ to look up the *types of locations* in t instead of calculating them from the values in μ

$$\frac{\Sigma(l) = \mathrm{T}_1}{\Gamma \mid \Sigma \vdash l : \mathtt{Ref}\ \mathrm{T}_1} \qquad \text{(T-Loc)}$$

i.e., *typing* is now *a four-place relation on* contexts, *store typings*, terms, and types.

Proviso: the typing rules ***accurately predict*** the results of evaluation *only if* the *concrete store* used during evaluation actually *conforms to* the store typing

Final typing rules

$$\frac{\Sigma(l) = \mathtt{T_1}}{\Gamma \mid \Sigma \vdash l : \mathtt{Ref\ T_1}} \qquad \text{(T-Loc)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t_1} : \mathtt{T_1}}{\Gamma \mid \Sigma \vdash \mathtt{ref\ t_1} : \mathtt{Ref\ T_1}} \qquad \text{(T-Ref)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t_1} : \mathtt{Ref\ T_{11}}}{\Gamma \mid \Sigma \vdash \mathtt{!t_1} : \mathtt{T_{11}}} \qquad \text{(T-Deref)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t_1} : \mathtt{Ref\ T_{11}} \qquad \Gamma \mid \Sigma \vdash \mathtt{t_2} : \mathtt{T_{11}}}{\Gamma \mid \Sigma \vdash \mathtt{t_1 := t_2} : \mathtt{Unit}} \qquad \text{(T-Assign)}$$

Store Typing

Where do *these store typings* come from?

When we first typecheck a program, there will be *no explicit locations*, so we can use *an empty store typing*, since the locations arise only in terms that are *the intermediate results* of evaluation

So, when a new location is created during evaluation,

$$\frac{l \notin dom(\mu)}{\texttt{ref}\ v_1 \mid \mu \longrightarrow l \mid (\mu, l \mapsto v_1)} \qquad \text{(E-RefV)}$$

we can observe the type of v_1 and *extend* the "*current store typing*" appropriately.

Store Typing

As evaluation proceeds and *new locations are created*, *the store typing is extended* by looking at the type of the initial values being placed in newly allocated cells

Σ only records the *association*

between

already-allocated storage cells and

their types

Safety

Coherence between the statics and the dynamics

Well-formed programs are well-behaved

when executed

Preservation

the steps of evaluation

preserve typing

Preservation

How to express the statement of preservation?

First attempt: just add *stores* and *store typings* in the appropriate places

Theorem(?): if $\Gamma \mid \Sigma \vdash t : T$ and $t|\mu \longrightarrow t'|\mu'$,

then $\Gamma \mid \Sigma \vdash t' : T$

Right??

Wrong! Why ?

Because Σ and μ here are not constrained to have anything to do with each other!

Exercise: Construct an example that breaks this statement of preservation

Preservation

Definition: A store μ is said to be *well typed* with respect to a typing context Γ and a store typing Σ, written $\Gamma \mid \Sigma \vdash \mu$, if $dom(\mu) = dom(\Sigma)$ and $\Gamma \mid \Sigma \vdash \mu(l)\colon\ \Sigma(l)$ for every $\mathrm{l} \in dom(\mu)$

Theorem (?) : if

$$\Gamma \mid \Sigma \vdash \mathrm{t} : \mathrm{T}$$

$$\mathrm{t} \mid \mu \longrightarrow \mathrm{t}' \mid \mu'$$

$$\Gamma \mid \Sigma \vdash \mu$$

then $\Gamma \mid \Sigma \vdash \mathrm{t}' : \mathrm{T}$

Right this time?

Still wrong !

Why? Where? (E-REFV） 13.5.2

Preservation

Creation of a *new reference cell* ...

$$\frac{l \notin dom(\mu)}{\mathrm{ref}\ \mathrm{v}_1 \,|\, \mu \longrightarrow l \,|\, (\mu, l \mapsto \mathrm{v}_1)} \qquad \text{(E-RefV)}$$

... *breaks the correspondence* between the store typing and the store.

Since *the store can grow* *during evaluation*:

Creation of a new reference cell yields a store with a *larger domain* than the initial one, making the conclusion *incorrect*: if μ' includes a binding for ***a fresh location*** l , then l cann't be in the domain of Σ , and it will not be the case that $\boldsymbol{t'}$ ***is typable under*** $\boldsymbol{\Sigma}$

Preservation

Theorem: if

$\Gamma \mid \Sigma \vdash \mathrm{t} : \mathrm{T}$

$\Gamma \mid \Sigma \vdash \mu$

$\mathrm{t} \mid \mu \longrightarrow \mathrm{t}' \mid \mu'$

then, for *some* $\Sigma' \supseteq \Sigma$,

$\Gamma \mid \Sigma' \vdash \mathrm{t}' : \mathrm{T}$

$\Gamma \mid \Sigma' \vdash \mu'$.

A correct version.

What is Σ' ?

Proof: Easy extension of the preservation proof for $\lambda_{\rightarrow}$

Progress

well-typed expressions are
either *values*
or can be *further evaluated*

Progress

Theorem:

Suppose t is a closed, well-typed term

(i.e., $\Gamma | \Sigma \vdash \mathrm{t} : \mathrm{T}$ for some T and Σ)

then either t is a *value* or else, for any store μ such that $\Gamma | \Sigma \vdash \mu$, there is some term t' and store μ' with

$$\mathrm{t} \mid \mu \longrightarrow \mathrm{t}' \mid \mu'$$

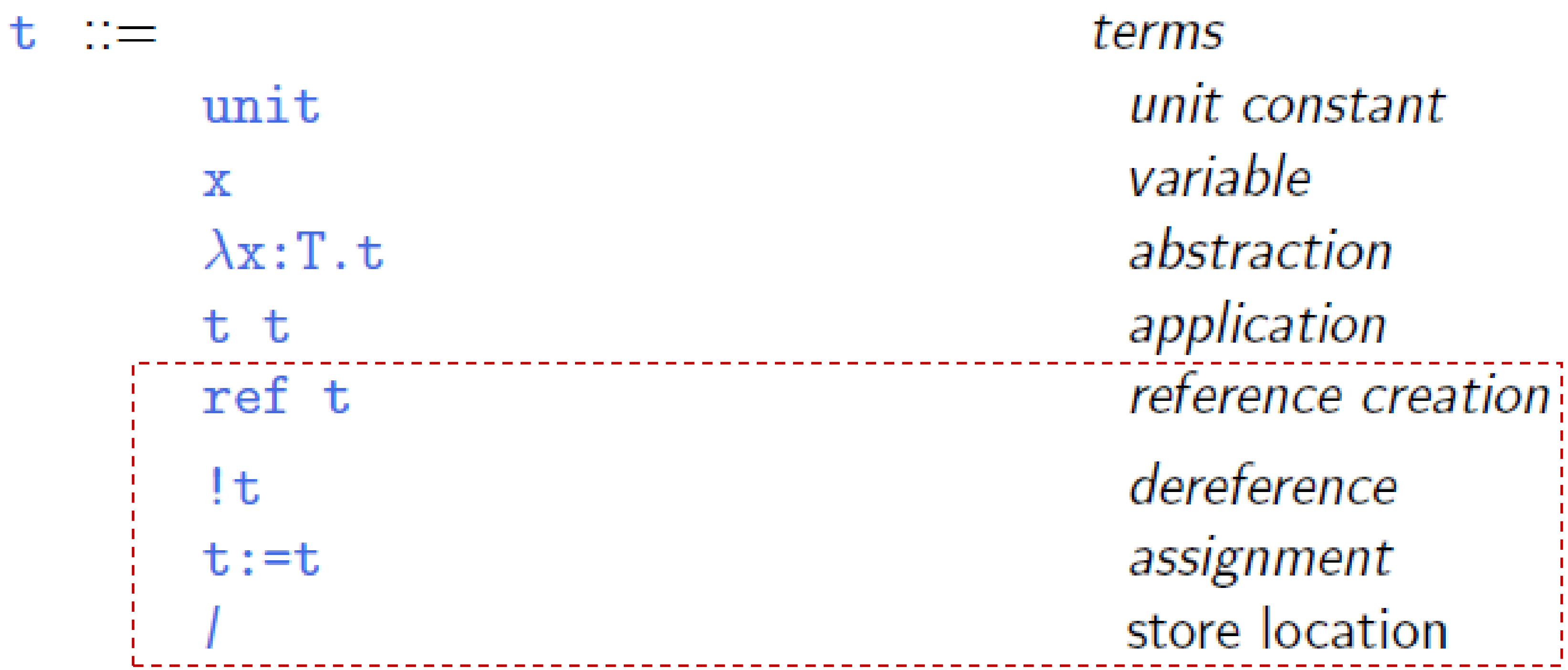
Safety

- preservation and progress together constitute the proof of safety
  - progress theorem ensures that well-typed expressions don’t get stuck in an ill-defined state, and
  - preservation theorem ensures that if a step is a taken the result remains well-typed (*with the same type*).
- These two parts ensure the *statics and dynamics* are coherent, and that no ill-defined states can ever be encountered while evaluating a well-typed expression

In summary …

Syntax

We added to $\lambda_{\to}$ (with Unit) syntactic forms for *creating*, *dereferencing*, and *assigning* reference cells, plus a new type constructor Ref.



`t ::=`	*terms*
`unit`	*unit constant*
`x`	*variable*
`λx:T.t`	*abstraction*
`t t`	*application*
`ref t`	*reference creation*
`!t`	*dereference*
`t:=t`	*assignment*
l	store location

Evaluation

Evaluation relation: $\mathsf{t} \mid \mu \longrightarrow \mathsf{t}' \mid \mu'$

$$\frac{l \notin dom(\mu)}{\texttt{ref}\ \ \mathtt{v}_1 \mid \mu \longrightarrow l \mid (\mu, l \mapsto \mathtt{v}_1)} \qquad (\text{E-}\textsc{RefV})$$

$$\frac{\mu(l) = \mathtt{v}}{\texttt{!}\, l \mid \mu \longrightarrow \mathtt{v} \mid \mu} \qquad (\text{E-}\textsc{DerefLoc})$$

$$l \texttt{:=} \mathtt{v}_2 \mid \mu \longrightarrow \texttt{unit} \mid [l \mapsto \mathtt{v}_2]\mu \qquad (\text{E-}\textsc{Assign})$$

Typing

Typing becomes *a* *four-place* relation: $\Gamma \mid \Sigma \vdash \mathtt{t} : \mathtt{T}$

$$\frac{\Sigma(l) = \mathtt{T}_1}{\Gamma \mid \Sigma \vdash l : \mathtt{Ref}\ \mathtt{T}_1} \quad \text{(T-Loc)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t}_1 : \mathtt{T}_1}{\Gamma \mid \Sigma \vdash \mathtt{ref}\ \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_1} \quad \text{(T-Ref)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_{11}}{\Gamma \mid \Sigma \vdash \mathtt{!t}_1 : \mathtt{T}_{11}} \quad \text{(T-Deref)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_{11} \qquad \Gamma \mid \Sigma \vdash \mathtt{t}_2 : \mathtt{T}_{11}}{\Gamma \mid \Sigma \vdash \mathtt{t}_1 \mathtt{:=} \mathtt{t}_2 : \mathtt{Unit}} \quad \text{(T-Assign)}$$

Preservation

Theorem: if

$\Gamma \mid \Sigma \vdash t : T$

$\Gamma \mid \Sigma \vdash \mu$

$t \mid \mu \longrightarrow t' \mid \mu'$

then, for some $\Sigma' \supseteq \Sigma$,

$\Gamma \mid \Sigma' \vdash t' : T$

$\Gamma \mid \Sigma' \vdash \mu'$.

Progress

Theorem: Suppose t is a *closed, well-typed* term (that is, $\emptyset \mid \Sigma \vdash \mathrm{t} : \mathrm{T}$ for some T and Σ). Then either t is a value or else, for any store μ such that $\emptyset \mid \Sigma \vdash \mu$, there is some term t' and store μ' with $\mathrm{t} \mid \mu \longrightarrow \mathrm{t}' \mid \mu'$

Others …

Arrays

Fix-sized vectors of values. All of the values must have the *same type*, and the fields in the array can be accessed and modified.

e.g., arrays can be created with in Ocaml

[|e_1; … ; e_n|]

```
# let a = [|1;3;5;7;9|];;
val a : int array = [|1;3;5;7;9|]
#a;;
-: int array = [|1;3;5;7;9|]
```

Arrays

```
let  f a =
   for i = 1 to Array.length a - 1 do
       let val_i = a.(i) in
       let j = ref i in
       while !j > 0 && val_i < a.(!j - 1) do
          a.(!j) <- a.(!j - 1);
          j := !j - 1
      done;
      a.(!j) <- val_i
  done;;
```

Recursion via references

Indeed, we can define *arbitrary recursive functions* using references

1. Allocate a ref cell and initialize it with a *dummy function* of the appropriate type:

$$\mathrm{fact}_{ref} \;=\; \mathrm{ref}\,(\lambda \mathrm{n}{:}\,\mathrm{Nat}.\,0)$$

2. Define *the body of the function* we are interested in, using *the contents of the reference cell* for making recursive calls:

$$\mathrm{fact}_{body} =$$
$$\lambda \mathrm{n}{:}\,\mathrm{Nat}.$$
$$\text{if iszero n then 1 else times n } ((!\,\mathrm{fact}_{ref})(\mathrm{pred\ n}))$$

3. “Backpatch” by storing the real body into the reference cell:

$$\mathrm{fact}_{ref} := \mathrm{fact}_{body}$$

4. Extract the contents of the reference cell and use it as desired:

$$\mathrm{fact} \;=\; !\,\mathrm{fact}_{ref}$$

Homework☺

- Read chapter 13
- Read and chew over the codes of *fullref*.

- HW: 13.4.1 and 13.5.8

- Preview chapter 14