Design Principles of Programming Languages
编程语言的设计原理

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Type Inference
类型推导

Type Erasure & Inference for System F

$$erase(x) \stackrel{\text{def}}{=} x$$

$$erase(\lambda x{:}T_1 . t_2) \stackrel{\text{def}}{=} \lambda x. erase(t_2)$$

$$erase(t_1\ t_2) \stackrel{\text{def}}{=} erase(t_1)\ erase(t_2)$$

$$erase(\lambda X. t_2) \stackrel{\text{def}}{=} erase(t_2)$$

$$erase(t_1\ [T_2]) \stackrel{\text{def}}{=} erase(t_1)$$

Definition (Type Inference)

Given an untyped term m, whether we can find some well-typed term t such that $erase(t) = m$.

Theorem (Wells, 1994[1])

Type inference for System F is **undecidable**.

[1] J. B. Wells. 1994. Typability and Type Checking in the Second-Order λ-Calculus Are Equivalent and Undecidable. In *Logic in Computer Science* (LICS'94), 176–185. DOI: 10.1109/LICS.1994.316068.

Partial Erasure & Inference for System F

$$
\begin{aligned}
erase_p(x) &\stackrel{\text{def}}{=} x \\
erase_p(\lambda x{:}T_1 .\, t_2) &\stackrel{\text{def}}{=} \lambda x{:}T_1 .\, erase_p(t_2) \\
erase_p(t_1\ t_2) &\stackrel{\text{def}}{=} erase_p(t_1)\ erase_p(t_2) \\
erase_p(\lambda X.\, t_2) &\stackrel{\text{def}}{=} \lambda X.\, erase_p(t_2) \\
erase_p(t_1\ [T_2]) &\stackrel{\text{def}}{=} erase_p(t_1)\ [\,]
\end{aligned}
$$

Theorem (Boehm 1985[2], 1989[3])

It is **undecidable** whether, given a closed term s in which type applications are marked but the arguments are omitted, there is some well-typed System-F term t such that $erase_p(t) = s$.

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[2] H.-J. Boehm. 1985. Partial Polymorphic Type Inference is Undecidable. In *Symp. on Foundations of Computer Science* (SFCS'85), 339–345. DOI: 10.1109/SFCS.1985.44.

[3] H.-J. Boehm. 1989. Type Inference in the Presence of Type Abstraction. In *Prog. Lang. Design and Impl.* (PLDI'89), 192–206. DOI: 10.1145/73141.74835.

Fragments of System F

Prenex Polymorphism

- Type variables range only over quantifier-free types (**monotypes**).
- Quantified types (**polytypes**) are not allows to appear on the left-hand sides of arrows.

Rank-2 Polymorphism

A type is said to be of rank 2 if no path from its root to a $\forall$ quantifier passes to the left of 2 or more arrows.

$$(\forall X.X \to X) \to \texttt{Nat} \quad ✓$$

$$\texttt{Nat} \to ((\forall X.X \to X) \to (\texttt{Nat} \to \texttt{Nat})) \quad ✓$$

$$((\forall X.X \to X) \to \texttt{Nat}) \to \texttt{Nat} \quad ✗$$

Remark

Prenex polymorphism is a **predicative** and rank-1 fragment of System F.
Type inference for ranks 2 and lower is **decidable**!

Simply-Typed Lambda-Calculus with Type Variables

Syntax

$$
\begin{aligned}
t &::= x \mid \lambda x{:}T.\, t \mid t\ t \mid \ldots \\
v &::= \lambda x{:}T.\, t \mid \ldots \\
T &::= X \mid T \to T \mid \ldots \\
\Gamma &::= \varnothing \mid \Gamma, x : T
\end{aligned}
$$

Typing

$$
\frac{x : T \in \Gamma}{\Gamma \vdash x : T}\ \text{T-Var}
\qquad
\frac{\Gamma, x : T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x{:}T_1.\, t_2 : T_1 \to T_2}\ \text{T-Abs}
\qquad
\frac{\Gamma \vdash t_1 : T_{11} \to T_{12} \qquad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}}\ \text{T-App}
$$

Type Substitutions

Definition

A type substitution is a finite mapping from type variables to types.

Example

We define $\sigma \stackrel{\text{def}}{=} [X \mapsto \texttt{Bool}, Y \mapsto U]$ for the substitution that maps X to $\texttt{Bool}$ and Y to U.
We write $dom(\cdot)$ for left-hand sides of pairs in a substitution, e.g., $dom(\sigma) = \{X, Y\}$.
We write $range(\cdot)$ for the right-hand sides of pairs in a substitution, e.g., $range(\sigma) = \{\texttt{Bool}, U\}$.

Remark

The pairs of a substitution are applied **simultaneously**.
For example, $[X \mapsto \texttt{Bool}, Y \mapsto X \to X]$ maps Y to $X \to X$, not $\texttt{Bool} \to \texttt{Bool}$.

Type Substitutions

Application of a Substitution to Types

$$\sigma(X) \stackrel{\text{def}}{=} \begin{cases} T & \text{if } (X \mapsto T) \in \sigma \\ X & \text{if } X \text{ is not in the domain of } \sigma \end{cases}$$

$$\sigma(\texttt{Nat}) \stackrel{\text{def}}{=} \texttt{Nat}$$

$$\sigma(\texttt{Bool}) \stackrel{\text{def}}{=} \texttt{Bool}$$

$$\sigma(T_1 \to T_2) \stackrel{\text{def}}{=} \sigma(T_1) \to \sigma(T_2)$$

Composition of Substitutions

$$\sigma \circ \gamma \stackrel{\text{def}}{=} \left[\begin{array}{ll} X \mapsto \sigma(T) & \text{for each } (X \mapsto T) \in \gamma \\ X \mapsto T & \text{for each } (X \mapsto T) \in \sigma \text{ with } X \notin dom(\gamma) \end{array} \right]$$

Type Substitutions

Application of a Substitution to Contexts

$$\sigma(x_1 : T_1, \ldots, x_n : T_n) \stackrel{\text{def}}{=} (x_1 : \sigma(T_1), \ldots, x_n : \sigma(T_n))$$

Application of a Substitution to Terms

$$\sigma(x) \stackrel{\text{def}}{=} x$$

$$\sigma(\lambda x{:}T_1 .\, t_2) \stackrel{\text{def}}{=} \lambda x{:}\sigma(T_1) .\, \sigma(t_2)$$

$$\sigma(t_1\ t_2) \stackrel{\text{def}}{=} \sigma(t_1)\ \sigma(t_2)$$

Theorem (Preservation of Typing under a Substitution)

If σ is any type substitution and $\Gamma \vdash t : T$, then $\sigma(\Gamma) \vdash \sigma(t) : \sigma(T)$.

Type Inference

Definition (Type Inference in terms of Substitutions)

Let Γ be a context and t be a term. **A solution for** (Γ, t) is a pair (σ, T) such that $\sigma(\Gamma) \vdash \sigma(t) : T$.

Remark (Two Views of $\sigma(\Gamma) \vdash \sigma(t) : T$)

- **Type Infernece**: does there exist **some** σ such that $\sigma(\Gamma) \vdash \sigma(t) : T$ for some T?
- Another view: for **every** σ, do we have $\sigma(\Gamma) \vdash \sigma(t) : T$ for some T?
  - This corresponds to **parametric polymorphism**, e.g., $\varnothing \vdash \lambda f{:}X \to X.\, \lambda a{:}X.\, f\ (f\ a) : (X \to X) \to X \to X$.

Example

Let $\Gamma \stackrel{\text{def}}{=} f : X, a : Y$ and $t \stackrel{\text{def}}{=} f\ a$. Below gives some solutions for (Γ, t):

σ	T	σ	T
$[X \mapsto Y \to \mathsf{Nat}]$	Nat	$[X \mapsto Y \to Z]$	Z
$[X \mapsto Y \to Z, Z \mapsto \mathsf{Nat}]$	Z	$[X \mapsto Y \to \mathsf{Nat} \to \mathsf{Nat}]$	$\mathsf{Nat} \to \mathsf{Nat}$
$[X \mapsto \mathsf{Nat} \to \mathsf{Nat}, Y \mapsto \mathsf{Nat}]$	Nat		

Erasure (revisited)

$$erase(x) \stackrel{\text{def}}{=} x$$

$$erase(\lambda x{:}T_1 .\, t_2) \stackrel{\text{def}}{=} \lambda x.\, erase(t_2)$$

$$erase(t_1\ t_2) \stackrel{\text{def}}{=} erase(t_1)\ erase(t_2)$$

Definition (Type Inference)

Let Γ be a context and m be an untyped term. A solution for (Γ, m) is a substitution (σ, T) such that $\sigma(\Gamma) \vdash m : T$.

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T}\ \text{T-Var} \qquad \frac{\Gamma, x : T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x.\, t_2 : T_1 \to T_2}\ \text{T-Abs} \qquad \frac{\Gamma \vdash t_1 : T_{11} \to T_{12} \qquad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}}\ \text{T-App}$$

Given the derivation, it is trivial to construct a well-typed term t such that $erase(t) = m$.

Constraint Typing

Definition

A constraint set C is a set of equations $\{S_i = T_i{}^{1\ldots n}\}$ where S_i's and T_i's are types.

$\Gamma \vdash t : T \mid_{\mathcal{X}} C$: "term t has type T under context Γ whenever constraints C are satisfied"

The set $\mathcal{X}$ is used to track **new** type variables introduced in each subderivation.

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T \mid_{\varnothing} \{\}} \text{ CT-Var} \qquad \frac{\Gamma, x : T_1 \vdash t_2 : T_2 \mid_{\mathcal{X}} C}{\Gamma \vdash \lambda x{:}T_1 .\, t_2 : T_1 \to T_2 \mid_{\mathcal{X}} C} \text{ CT-Abs}$$

$$\frac{\begin{array}{c}\Gamma \vdash t_1 : T_1 \mid_{\mathcal{X}_1} C_1 \qquad \Gamma \vdash t_2 : T_2 \mid_{\mathcal{X}_2} C_2 \qquad \mathcal{X}_1 \cap \mathcal{X}_2 = \mathcal{X}_1 \cap FV(T_2) = \mathcal{X}_2 \cap FV(T_1) = \varnothing \\ X \notin \mathcal{X}_1, \mathcal{X}_2, T_1, T_2, C_1, C_2, \Gamma, t_1, t_2 \qquad C' = C_1 \cup C_2 \cup \{T_1 = T_2 \to X\}\end{array}}{\Gamma \vdash t_1\ t_2 : X \mid_{\mathcal{X}_1 \cup \mathcal{X}_2 \cup \{X\}} C'} \text{ CT-App}$$

Question (Exercise 22.3.3)

Construct a constraint typing derivation for $\lambda x{:}X.\ \lambda y{:}Y.\ \lambda z{:}Z.\ (x\ z)\ (y\ z)$.

Solutions for Constraint Typing

Definition

A substitution σ is said to **unify** an equation $S = T$ if $\sigma(S) = \sigma(T)$.
We say that σ unifies a constraint set C if it unifies every equation in C.

Definition

Suppose that $\Gamma \vdash t : S \mid_{\mathcal{X}} C$. **A solution for** (Γ, t, S, C) is a pair (σ, T) such that σ unified C and $\sigma(S) = T$.

Remark

Recall that **a solution for** (Γ, t) is a pair (σ, T) such that $\sigma(\Gamma) \vdash \sigma(t) : T$.
What are the relation between the two definitions of solutions for type inference?

Properties of Constraint Typing

Theorem (Soundness)

Suppose that $\Gamma \vdash t : S \mid C$. If (σ, T) is a solution for (Γ, t, S, C), then it is also a solution for (Γ, t).

Proof Sketch

By induction on the derivation of constraint typing.

Theorem (Completeness)

Suppose $\Gamma \vdash t : S \mid_{\mathcal{X}} C$. If (σ, T) is a solution for (Γ, t) and $dom(\sigma) \cap \mathcal{X} = \varnothing$, then there is some solution (σ', T) for (Γ, t, S, C) such that $\sigma' \setminus \mathcal{X} = \sigma$.

Proof Sketch

By induction on the derivation of constraint typing.

Unification

Remark

Hindley (1969)[4] and Milner (1978)[5] apply unification to calculate **a "best" solution** to a given constraint set.

Definition

A substitution σ is less specific (or **more general**) than a substitution σ', written $\sigma \sqsubseteq \sigma'$, if $\sigma' = \gamma \circ \sigma$ for some γ.

A **principal unifier** (or sometimes **most general unifier**) for a constraint set C is a substitution σ that unifies C and such that $\sigma \sqsubseteq \sigma'$ for every substitution σ' unifying C.

Question (Exercise 22.4.3)

Write down principal unifiers (when they exist) for the following sets of constraints:

$$\{X = \mathsf{Nat}, Y = X \to X\} \quad \{\mathsf{Nat} \to \mathsf{Nat} = X \to Y\} \quad \{X \to Y = Y \to Z, Z = U \to W\}$$

$$\{\mathsf{Nat} = \mathsf{Nat} \to Y\} \quad \{Y = \mathsf{Nat} \to Y\} \quad \{\}$$

[4] R. Hindley. 1969. The Principal Type-Scheme of an Object in Combinatory Logic. *Trans. of the American Math. Society*, 146, 29–60. doi: 10.2307/1995158.

[5] R. Milner. 1978. A Theory of Type Polymorphism in Programming. *J. Comput. Syst. Sci.*, 17, 348–375, 3. doi: 10.1016/0022-0000(78)90014-4.

Unification Algorithm

$$
\begin{aligned}
unify(C) \quad = \quad & \text{if } C = \varnothing, \text{ then } [\,] \\
& \text{else let } \{S = T\} \cup C' = C \text{ in} \\
& \quad \text{if } S = T \\
& \quad\quad \text{then } unify(C') \\
& \quad \text{else if } S = X \text{ and } X \notin FV(T) \\
& \quad\quad \text{then } unify([X \mapsto T]C') \circ [X \mapsto T] \\
& \quad \text{else if } T = X \text{ and } X \notin FV(S) \\
& \quad\quad \text{then } unify([X \mapsto S]C') \circ [X \mapsto S] \\
& \quad \text{else if } S = S_1 \to S_2 \text{ and } T = T_1 \to T_2 \\
& \quad\quad \text{then } unify(C' \cup \{S_1 = T_1, S_2 = T_2\}) \\
& \quad \text{else} \\
& \quad\quad fail
\end{aligned}
$$

What if we omit the occur checks (i.e., $X \notin FV(T)$ and $X \notin FV(S)$)?

Correctness of Unification Algorithm

Theorem

The algorithm unify always terminates, failing when given a non-unifiable constraint set as input and otherwise returning a principal unifier.

Proof Sketch

- **Termination**: define the **degree** of C to be the pair (number of distinct type variables, total size of types).
- $\mathit{unify}(C)$ **returns a unifier**: prove by induction on the number of recursive calls to unify.
  - Fact: if σ unifies $[\mathrm{X} \mapsto \mathrm{T}]D$, then $\sigma \circ [\mathrm{X} \mapsto \mathrm{T}]$ unifies $\{\mathrm{X} = \mathrm{T}\} \cup D$.
- $\mathit{unify}(C)$ returns a **principal** unifier: prove by induction on the number of recursive calls.

Principal Types

Definition

A principal solution for (Γ, t, S, C) is a solution (σ, T) such that, $\sigma \sqsubseteq \sigma'$ for any other solution (σ', T').
When (σ, T) is a principal solution, we call T **a principal type** of t under Γ.

Theorem

If (Γ, t, S, C) has any solution, then it has a principal one.
The *unify* algorithm can be used to determine if there exists a solution and, if so, to calculate a principal one.

Corollary

It is decidable whether (Γ, t) has a solution.

Remark

Recall that type inference for System F is **undecidable**.

Recall: Prenex Polymorphism

Prenex Polymorphism

- Type variables range only over quantifier-free types (**monotypes**).
- Quantified types (**polytypes**) are not allows to appear on the left-hand sides of arrows.

Let-Polymorphism is a Variant of Prenex Polymorphism where …

- Quantifiers can only occur at the outermost level of types.
- Type abstractions are implicitly introduced at **let-bindings**.
- Type applications are implicitly introduced at **variables**.

Let-Polymorphism as a Fragment of System F

Syntax

$$
\begin{aligned}
t &::= x \mid \lambda x{:}T.\, t \mid t\ t \mid \texttt{let}\ x = t\ \texttt{in}\ t \mid \ldots \\
v &::= \lambda x{:}T.\, t \mid \ldots \\
T &::= X \mid T \to T \mid \ldots \\
\mathbb{T} &::= \forall X_1 \ldots X_n.\, T \\
\Gamma &::= \varnothing \mid \Gamma, x : \mathbb{T}
\end{aligned}
$$

Typing

$$
\frac{\Gamma \vdash t_1 : T_1 \qquad \{X_1, \ldots, X_n\} = FV(T_1) \setminus FV(\Gamma) \qquad \mathbb{T}_1 = \forall X_1 \ldots X_n.\, T_1 \qquad \Gamma, x : \mathbb{T}_1 \vdash t_2 : T_2}{\Gamma \vdash \texttt{let}\ x = t_1\ \texttt{in}\ t_2 : T_2}\ \text{T-Let}
$$

$$
\frac{x : \forall X_1 \ldots X_n.\, T \in \Gamma}{\Gamma \vdash x : [X_1 \mapsto S_1, \ldots, X_n \mapsto S_n]T}\ \text{T-Var}
$$

Let-Polymorphism as a Fragment of System F

Example

```
let double = λf:(X→X). λa:X. f (f a) in
  {double (λx:Nat. succ (succ x)) 1,
   double (λx:Bool. x) false}
```

(T-Let): $\forall X. (X \to X) \to X \to X$

(T-Var): $(\mathtt{Nat} \to \mathtt{Nat}) \to \mathtt{Nat} \to \mathtt{Nat}$

(T-Var): $(\mathtt{Bool} \to \mathtt{Bool}) \to \mathtt{Bool} \to \mathtt{Bool}$

Observation

Let-polymorphism can be equivalently implemented in simply-typed lambda-calculus with the following rule:

$$\frac{\Gamma \vdash t_1 : T_1 \qquad \Gamma \vdash [x \mapsto t_1]t_2 : T_2}{\Gamma \vdash \mathtt{let}\; x = t_1 \;\mathtt{in}\; t_2 : T_2} \text{ T-LetPoly}$$

Constraint Typing for Let-Polymorphism

$$\frac{\Gamma \vdash t_1 : T_1 \mid_{\mathcal{X}_1} C_1 \qquad \{X_1,\ldots,X_n\} = FV(T_1) \cup FV(C_1) \setminus FV(\Gamma) \qquad \mathbb{T}_1 = \forall X_1 \ldots X_n.\, C_1 \supset T_1 \qquad \Gamma, x : \mathbb{T}_1 \vdash t_2 : T_2 \mid_{\mathcal{X}_2} C_2}{\Gamma \vdash \texttt{let}\; x = t_1 \;\texttt{in}\; t_2 : T_2 \mid_{\mathcal{X}_1 \cup \mathcal{X}_2} C_1 \cup C_2} \text{ CT-Let}$$

$$\frac{x : \forall X_1 \ldots X_n.\, C \supset T \in \Gamma \qquad Y_1,\ldots,Y_n \notin X_1,\ldots,X_n, T, \Gamma}{\Gamma \vdash x : [X_1 \mapsto Y_1,\ldots,X_n \mapsto Y_n]T \mid_{\{Y_1,\ldots,Y_n\}} [X_1 \mapsto Y_1,\ldots,X_n \mapsto Y_n]C} \text{ CT-Var}$$

Example

```
let double = λf:(X→X). λa:X. f (f a) in
```

(CT-Let): $\forall X, X_1, X_2. \{X \to X = X \to X_1, X \to X = X_1 \to X_2\} \supset (X \to X) \to X \to X_2 \mid \{\ldots\}$

```
  {double (λx:Nat. succ (succ x)) 1,
```

(CT-Var): $(Y \to Y) \to Y \to Y_2 \mid \{Y \to Y = Y \to Y_1, Y \to Y = Y_1 \to Y_2\} \cup \{Y \to Y = \mathsf{Nat} \to \mathsf{Nat}\}$

```
   double (λx:Bool. x) false}
```

(CT-Var): $(Z \to Z) \to Z \to Z_2 \mid \{Z \to Z = Z \to Z_1, Z \to Z = Z_1 \to Z_2\} \cup \{Z \to Z = \mathsf{Bool} \to \mathsf{Bool}\}$

Interaction with Side Effects

Example

Let-polymorphism would assign $\forall X.\,\texttt{Ref}(X \to X)$ to `r` in the following code:

```
let r = ref (λx:X. x) in
(r := (λx:Nat. succ x);
 (!r)true);
```

When type-checking the second line, we instantiate `r` to have type $\texttt{Ref}(\texttt{Nat} \to \texttt{Nat})$.
When type-checking the third line, we instantiate `r` to have type $\texttt{Ref}(\texttt{Bool} \to \texttt{Bool})$.
But this is **unsound**!

Value Restriction

A let-binding can be treated polymorphically—i.e., its free type variables can be generalized—only if its right-hand side is a **syntactic value**.

Homework

Question

Consider the following lambda-abstraction:

```
λ x:X. x x
```

Construct a constraint typing derivation for it.
Is the constraint set unifiable?
What if removing the occur checks in the *unify* algorithm and allowing recursive types, as shown below?
What is the result of this *unify* algorithm?

$$
\begin{aligned}
\mathit{unify}(C) \quad = \quad & \ldots \\
& \text{else if } S = \mathrm{X} \text{ and } \mathrm{X} \notin FV(\mathrm{T}) \\
& \quad \text{then } \mathit{unify}([\mathrm{X} \mapsto \mathrm{T}]C') \circ [\mathrm{X} \mapsto \mathrm{T}] \\
& \text{else if } S = \mathrm{X} \text{ and } \mathrm{X} \in FV(\mathrm{T}) \\
& \quad \text{then } \mathit{unify}([\mathrm{X} \mapsto \mu\mathrm{X}.\,\mathrm{T}]C') \circ [\mathrm{X} \mapsto \mu\mathrm{X}.\,\mathrm{T}] \\
& \ldots
\end{aligned}
$$