编程语言的设计原理
Design Principles of Programming Languages

2c4r

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Recap: untyped lambda-calculus

Syntax

t	::=	*terms:*
	x	*variable*
	λx.t	*abstraction*
	t t	*application*
v	::=	*values:*
	λx.t	*abstraction value*

Evaluation $t \longrightarrow t'$

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2} \quad \text{(E-App1)}$$

$$\frac{t_2 \longrightarrow t_2'}{v_1\ t_2 \longrightarrow v_1\ t_2'} \quad \text{(E-App2)}$$

$$(\lambda x.t_{12})\ v_2 \longrightarrow [x \mapsto v_2]t_{12} \quad \text{(E-AppAbs)}$$

- The *λ-calculus* embodies this kind of *function definition* and *application* in the purest possible form
  - terms in the pure *λ-calculus* are often called *λ-terms*
  - terms of the form *λx. t* are called *λ-abstractions* or just abstractions

Syntax

- **Definition** [Terms]:

  Let $\mathcal{V}$ be a *countable set* of variable names.

  The set of terms is *the smallest set* $\mathcal{T}$ such that

  1. $x \in \mathcal{T}$ for every $x \in \mathcal{V}$;
  2. if $t_1 \in \mathcal{T}$ and $x \in \mathcal{V}$, then $\lambda x.t_1 \in \mathcal{T}$;
  3. if $t_1 \in \mathcal{T}$ and $t_2 \in \mathcal{T}$, then $t_1\ t_2 \in \mathcal{T}$.

Syntactic conventions

- The *λ-calculus* provides *only one-argument functions*, all multi-argument functions must be written in curried style.
- The following *conventions* make the linear forms of terms easier to read and write:
  - **Application *binds more tightly*** *than abstraction*

    e.g., **λx. x y** means **λx. (x y)** not *(λx. x) y*
  - Application ***associates to the left***

    e.g., **t u v** means **(t u) v**, not *t (u v)*
  - Bodies of λ- abstractions *extend as far to the right as possible*

    e.g., **λx.λy.x y** means **λx. (λy. x y)**, not *λx. (λy. x) y*

Scope

- *An occurrence* of the variable x is said to be *bound* when it occurs in the body t of an abstraction $\lambda \mathrm{x.t}$, i.e.,
  - the λ-abstraction term $\lambda x.\mathrm{t}$ binds the variable x , and **the scope** of this binding is **the body** t.
  - λx is a *binder* (binding construct) whose *scope* is t.
    - e.g., $(\lambda \mathrm{x.}(\lambda \mathrm{y.x\ y}))\ \mathrm{y}$
  - a binder can be *renamed* as necessary
    - so-called: *alpha-renaming/ alpha-conversion*
    - e.g., $\lambda x.\mathrm{x} = \lambda \mathrm{y.\ y}$

Scope

- **Definition:** Free Variables of term t, written as FV(t):

$$\mathrm{FV}(x) = \{x\}$$

$$\mathrm{FV}(\lambda x.t_1) = \mathrm{FV}(t_1) \setminus \{x\}$$

$$\mathrm{FV}(t_1\ t_2) = \mathrm{FV}(t_1) \cup \mathrm{FV}(t_2)$$

Operational Semantics

- Computation rule

$$(\lambda x.t_{12})\ v_2 \longrightarrow [x \mapsto v_2]t_{12} \qquad \text{(E-AppAbs)}$$

- Congruence rules

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2} \qquad \text{(E-App1)}$$

$$\frac{t_2 \longrightarrow t_2'}{v_1\ t_2 \longrightarrow v_1\ t_2'} \qquad \text{(E-App2)}$$

Operational Semantics

- *Beta-reduction*: the only computation (substitution)

$$(\lambda x.\, t_{12})\; t_2 \longrightarrow [x \mapsto t_2]t_{12},$$

  - the term obtained by *replacing all free occurrences* of x in t_{12} by t_2
  - a *redex* (short for "*reducible expression*") : a term of the form *(λx.t) v* — a *λ-abstraction* applied to a *value*
  - the operation of rewriting a *redex* according to the above rule is called *beta-reduction*
- Examples:

$$(\lambda x.\, x)\; y \rightarrow\; y$$

$$(\lambda x.\, x\; (\lambda x.\, x))\; (u\; r) \rightarrow u\; r\; (\lambda x.\, x)$$

Substitution

$$
\begin{array}{lcll}
[x \mapsto s]x & = & s & \\
[x \mapsto s]y & = & y & \text{if } y \neq x \\
[x \mapsto s](\lambda y.t_1) & = & \lambda y.\,[x \mapsto s]t_1 & \text{if } y \neq x \text{ and } y \notin FV(s) \\
[x \mapsto s](t_1\ t_2) & = & [x \mapsto s]t_1\ [x \mapsto s]t_2 &
\end{array}
$$

Alpha-conversion : Terms that **differ only in the names of bound variables** are **interchangeable** in all contexts.

Example:

$[x \mapsto y\ z]\ (\lambda y.\ x\ y)$

$= [x \mapsto y\ z]\ (\lambda w.\ x\ w)$

$= \lambda w.\ y\ z\ w$

Bound Variables

- Recall that bound variables **can be renamed, at any moment,** to enable substitution:

$$
\begin{array}{lcll}
[x \mapsto s]x & = & s & \\
[x \mapsto s]y & = & y & \text{if } y \neq x \\
[x \mapsto s](\lambda y.t_1) & = & \lambda y.\,[x \mapsto s]t_1 & \text{if } y \neq x \text{ and } y \notin FV(s) \\
[x \mapsto s](t_1\ t_2) & = & [x \mapsto s]t_1\ [x \mapsto s]t_2 &
\end{array}
$$

- Variable Representation
  - Represent variables symbolically, with variable renaming mechanism
  - Represent variables symbolically, with bound variables are all different from each other and from any free variable
  - "Canonically" represent variables in a way such that renaming is unnecessary
  - No use of variables: combinatory logic, or something like Backus' functional language FP

Chapter 6
Nameless Representation of Terms

Terms and Contexts

Shifting and Substitution

Terms and Contexts

Nameless Terms

- *De Bruijn* idea: Replacing named variables by *natural numbers*, where the number k stands for "the variable bound by the $k'th$ enclosing λ".
  1. **Index Starts at 0 :** The count begins at 0 for the nearest enclosing lambda.
  2. **Each λ Adds a Level:** Traversing a lambda increases the binding depth by 1.
  3. **Count Upwards:** The index of a variable is the number of lambdas you encounter when moving up the term tree until you reach its binder.
- e.g.,
  - λx.x λ.0
  - λx.λy.x λ.λ. 1
  - λx.λy.y λ. λ. 0
  - λx.λy. x (y x) λ.λ. 1 (0 1)
  - (λx. x) (λy. y) (λ 0) (λ 0)

De Bruijn indices **vs** *De Bruijn terms/nameless terms*

Nameless Terms

- *De Bruijn* idea: Replacing named variables by *natural numbers*, where the number k stands for "the variable bound by the $k'th$ enclosing λ".

- *e.g.*, the corresponding nameless term for the following:

  c0 = λs. λz. z;

  c2 = λs. λz. s (s z);

  plus = λm. λn. λs. λz. m s (n z s);

  fix = λf. (λx. f (λy. (x x) y)) (λx. f (λy. (x x) y));

  foo = (λx. (λx. x)) (λx. x);

Nameless Terms

- Need to keep track of how many free variables each term may contain.

Definition [Terms]: Let $\mathcal{T}$ be *the smallest family of sets* $\{\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots\}$ such that

1. $k \in \mathcal{T}_n$ whenever $0 \le k < n$;
2. if $t_1 \in \mathcal{T}_n$ and $n>0$, then $\lambda.t_1 \in \mathcal{T}_{n-1}$;
3. if $t_1 \in \mathcal{T}_n$ and $t_2 \in \mathcal{T}_n$, then $(t_1\ t_2) \in \mathcal{T}_n$.

- **Note:**
  - terms with **no free variables** are called the **0-term**s; 1-terms (one **free variables), …**
  - $\mathcal{T}_n$ are set of terms with **at most n** free variables, n-terms, numbered between 0 and n-1: a given element of $\mathcal{T}_n$ need not have free variables with all these numbers, or indeed any free variables at all. When **t** is closed, for example, it will be an element of $\mathcal{T}_n$ for every n.
  - two ordinary terms are *equivalent* modulo renaming of bound variables iff they have the same de Bruijn representation.

Name Context

How to represent

$$\lambda x.\ y\ x$$

as a nameless term?

Here y is free variable.

We know what to do with x, but **we cannot see the binder** for y, so it is *not clear how "far away"* it might be and we do not know *what number* to assign to it.

To deal with these terms containing free variables, we need the idea of a **naming context.**

Name Context

Definition: Suppose x_0 through x_n are variable names from ν. The naming context

$\Gamma = x_n, x_{n-1}, \ldots x_1, x_0$ assigns to each x_i the *de Bruijn index* i.

Note that the *rightmost variable* in the sequence is given the index *0*; this matches the way we count *λ binders* — ***from right to left*** — when converting a named term to nameless form.

Name Context

- We write **dom(Γ)** for the set $\{x_n, \ldots x_1, x_0\}$ of variable names mentioned in Γ

- e.g., $\Gamma = x \mapsto 4; y \mapsto 3; z \mapsto 2; a \mapsto 1; b \mapsto 0$, under this Γ, we have
  - x (y z) ?
    4 (3 2)
  - λw. y w
    λ. 4 0
  - λw. λa. x
    λ. λ. 6

Shifting and Substitution

How to define substitution $[k \mapsto s]\ t$?

Shifting

- Under the naming context Γ : $x \mapsto 1, z \mapsto 2$

  $[1 \mapsto 2\ (\lambda.\ 0)\ ]\ \lambda.2 \longrightarrow\ ?$

  i.e., $[\ \mathbf{x} \mapsto \mathbf{z}\ \mathbf{(\lambda w.\ w)}\ ]\ \mathbf{\lambda y.\ x} \longrightarrow ?$

- When a substitution **goes under a λ-abstraction**, as in $[1 \mapsto s](\lambda.2)$ (i.e.,$[x \mapsto s](\lambda y.\ x)$, assuming that 1 is the index of x in the outer context), *the context* in which the substitution is taking place becomes *one variable longer than the original*.
  - We need to *increment the indices* of the *free variables* in s so that they keep referring to *the same names in the new context* as they did before.
  - e.g., s = 2 (λ. 0), , i.e., s = z (λw. w), assuming 2 is the index of z in the outer context, we need to shift the 2 but not the 0

- Shifting is just the auxiliary operation: ***renumber the indices*** *of the* ***free variables*** in a term.

Shifting

DEFINITION [SHIFTING]: The d-place shift of a term $\mathtt{t}$ above cutoff c, written $\uparrow_c^d(\mathtt{t})$, is defined as follows:

$$
\begin{array}{lcl}
\uparrow_c^d(\mathtt{k}) & = & \begin{cases} \mathtt{k} & \text{if } k < c \\ \mathtt{k} + d & \text{if } k \geq c \end{cases} \\
\uparrow_c^d(\lambda.\mathtt{t}_1) & = & \lambda.\ \uparrow_{c+1}^d(\mathtt{t}_1) \\
\uparrow_c^d(\mathtt{t}_1\ \mathtt{t}_2) & = & \uparrow_c^d(\mathtt{t}_1)\ \uparrow_c^d(\mathtt{t}_2)
\end{array}
$$

We write $\uparrow^d(\mathtt{t})$ for $\uparrow_0^d(\mathtt{t})$. □

1. What is $\uparrow^2(\lambda.\lambda.\ 1\ (0\ 2))$?

2. What is $\uparrow^2(\lambda.\ 0\ 1\ (\lambda.\ 0\ 1\ 2))$?

Substitution

DEFINITION [SUBSTITUTION]: The substitution of a term s for variable number j in a term t, written $[j \mapsto s]t$, is defined as follows:

$$
\begin{array}{lcl}
[j \mapsto s]k & = & \begin{cases} s & \text{if } k = j \\ k & \text{otherwise} \end{cases} \\
[j \mapsto s](\lambda.t_1) & = & \lambda.[j{+}1 \mapsto \uparrow^1(s)]t_1 \\
[j \mapsto s](t_1\ t_2) & = & ([j \mapsto s]t_1\ [j \mapsto s]t_2)
\end{array}
$$

□

$$
\begin{array}{lcll}
[x \mapsto s]x & = & s & \\
[x \mapsto s]y & = & y & \text{if } y \neq x \\
[x \mapsto s](\lambda y.t_1) & = & \lambda y.\,[x \mapsto s]t_1 & \text{if } y \neq x \text{ and } y \notin FV(s) \\
[x \mapsto s](t_1\ t_2) & = & [x \mapsto s]t_1\ [x \mapsto s]t_2 &
\end{array}
$$

Evaluation

- To define the *evaluation relation* on nameless terms, the only thing we *need to change* (i.e., the only place where *variable names* are mentioned) is the *beta-reduction rule (computation rules),* while keep the other rules identical to what as Figure 5-3.

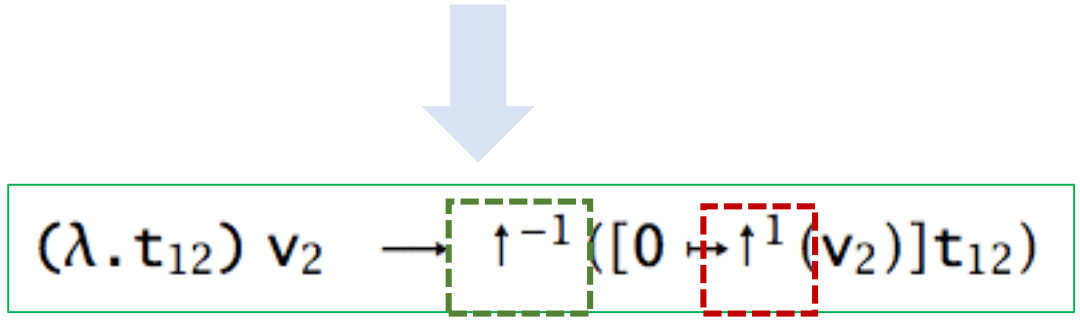
$$(\lambda x.\ t_{12})\ t_2 \longrightarrow [x \mapsto t_2]t_{12},$$

- How to change the above rule for nameless representation?

Evaluation

- Example:

$$(\lambda x.\, t_{12})\ t_2 \longrightarrow [x \mapsto t_2]t_{12},$$



$$(\lambda.t_{12})\ v_2 \longrightarrow \uparrow^{-1}([0 \mapsto \uparrow^{1}(v_2)]t_{12})$$

$$(\lambda.1\ 0\ 2)\ (\lambda.0) \longrightarrow 0\ (\lambda.0)\ 1$$

Homework

- Read Chapter 6.
  - Do Exercise 6.3.2.

> 6.3.2 EXERCISE [⋆⋆⋆]: De Bruijn's original article actually contained two different proposals for nameless representations of terms: the deBruijn *indices* presented here, which number lambda-binders "from the inside out," and *de Bruijn levels*, which number binders "from the outside in." For example, the term λx. (λy. x y) x is represented using deBruijn indices as λ. (λ. 1 0) 0 and using deBruijn levels as λ. (λ. 0 1) 0. Define this variant precisely and show that the representations of a term using indices and levels are isomorphic (i.e., each can be recovered uniquely from the other). □

- Read Chapter 7 and digest the *fulluntyped* implementation includes extensions such as numbers and booleans.